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Abstract: A natural generalization of graph colouring involves taking colours from a metric space and insisting that the endpoints of an edge receive colours separated by a minimum distance dictated by properties of the edge. In the q-backbone colouring problem, these minimum distances are either q or 1, depending on whether or not the edge is in the *backbone*. In this paper we consider the list version of this problem, with particular focus on colours in $\mathbb{Z}_p$ – this problem is closely related to the problem of circular choosability.

We first prove that the *list circular q-backbone chromatic number* of a graph is bounded by a function of the list chromatic number. We then consider the more general problem in which each edge is assigned an individual distance between its endpoints, and provide bounds using the Combinatorial Nullstellensatz. Through this result and through structural approaches, we achieve good bounds when both the graph and the backbone belong to restricted families of graphs.

Key-words: backbone colouring, list colouring, planar graph, combinatorial Nullenstellensatz

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Coloration dorsale circulaire sur listes

Résumé : Une généralisation naturelle de la coloration de graphe requiert de prendre des couleurs dans un espace métrique et que deux sommets reliés par une arête reçoivent des couleurs séparées par une distance minimum imposée par les propriétés de l'arête. Dans les cas de la coloration q-dorsale, ces distances minimum valent q ou 1, suivant que l'arête est dans la *dorsale* ou non. Dans ce rapport, nous considérons un version sur listes de ce problème, en prenant des couleurs dans $\mathbb{Z}_p$ – ce problème est étroitement lié au problème de la choisissabilité circulaire.

Nous prouvons d'abord que la *choisissabilité q-dorsale circulaire* d'un graphe est majorée par une fonction de la choisissabilité. Nous considérons ensuite un problème plus général dans lequel chaque arête est munie d'une distance propre requise entre ses extrémités et donnons des bornes supérieures à l'aide du Nullstellensatz combinatoire. A l'aide de ce résultat et d'approches structurelles, nous obtenons de bonnes bornes quand le graphe et la dorsale appartiennent à des classes particulières de graphes.

Mots-clés : coloration dorsale, coloration sur listes, graphe planaire, Nullenstellensatz combinatoire

1 Introduction

All graphs considered in this paper are simple. Let $G=(V,E)$ be a graph, and let $H=(V,E(H))$ be a subgraph of G, called the *backbone*. A *k-colouring* of G is a mapping $f: V \to \{1,2,\dots,k\}$. Let f be a k-colouring of G. It is a *proper colouring* if $|f(u)-f(v)| \geq 1$. It is a *q-backbone colouring* for (G,H) if f is a proper colouring of G and $|f(u)-f(v)| \geq q$ for all edges $uv \in E(H)$. The *chromatic number* $\chi(G)$ is the smallest integer k for which there exists a proper k-colouring of G. The *q-backbone chromatic number* $\mathrm{BBC}_q(G,H)$ is the smallest integer l for which there exists a q-backbone k-colouring of (G,H).

If f is a proper k-colouring of G, then g defined by $g(v) = q \cdot f(v) - (q-1)$ is a q-backbone $(q \cdot k - q + 1)$-colouring of (G,H) for any spanning subgraph H of G. Hence,

$$\mathrm{BBC}_q(G,H) \leq q \cdot \chi(G) - q + 1. \tag{1}$$

In [4, 5], Broersma et al. showed that for any integer k there is a graph G with a spanning tree T such that $\mathrm{BBC}_2(G,T) = 2k-1$.

One can generalize the notion of backbone colouring by allowing a more complicated structure of the colour space. A natural choice is to impose a circular metric on the colours. We can see $\mathbb{Z}_k$ as a cycle of length k with vertex set $\{1,\dots,k\}$ together with the graphical distance $||_k$. Then $|a-b|_k \geq q$ if and only if $q \leq |a-b| \leq k-q$. A *circular q-backbone k-colouring* of G or *q-backbone $\mathbb{Z}_k$-colouring* of (G,H) is a mapping $f: V(G) \to \mathbb{Z}_k$ such that $c(v) \neq c(u)$ for each edge $uv \in E(G)$ and $|c(u)-c(v)|_r \geq q$ for each edge $uv \in E(H)$. The *circular q-backbone chromatic number* of a graph pair (G,H), denoted $\mathrm{CBC}_q(G,H)$, is the minimum k such that (G,H) admits a circular q-backbone k-colouring.

A backbone $\mathbb{Z}_k$-colouring is trivially a backbone k-colouring. On the other hand, a backbone k-colouring yields a circular backbone $\mathbb{Z}_{k+q-1}$-colouring. Hence for every graph pair (G,H) (where H is a subgraph of G), we have

$$\mathrm{BBC}_q(G,H) \leq \mathrm{CBC}_q(G,H) \leq \mathrm{BBC}_q(G,H) + q - 1. \tag{2}$$

It is well-known that $\mathrm{CBC}_q(G,G) = q \cdot \chi(G)$ and so

$$\mathrm{CBC}_q(G,H) \leq q \cdot \chi(G) \quad \text{for every subgraph } H \text{ of } G. \tag{3}$$

This bound is tight if H contains a complete graph of size $\chi(G)$. In particular, it is the case when G is bipartite, and H is non-empty. On the other hand, Broersma et al. [6] gave better upper bounds when the backbone is a matching. We explore similar restrictions shortly, but first we define a list colouring analogue.

1.1 List backbone colourings

The notions of backbone and circular backbone colouring naturally generalize to list colouring. An *S-list assignment* of a graph G is a mapping L which assigns to each vertex $v \in V(G)$ a prescribed list of colours $L(v) \subseteq S$, where S is a set of colours. For a list assignment L, an *L-colouring* is a colouring c such that $c(v) \in L(v)$ for every vertex. The graph pair (G,H) is said to be *circularly q-backbone k-choosable* if for any $\mathbb{Z}_p$-list assignment L such that $|L(v)| \geq k$, there is an L-colouring that is a q-backbone $\mathbb{Z}_p$-colouring. The *list circular q-backbone number* of a pair (G,H), denoted $\mathrm{CBC}_q^\ell(G,H)$ is the least integer k such that (G,H) is circularly q-backbone k-choosable.

The concept of *circular choosability*, introduced by Mohar [13] and Zhu [18], is closely related to list circular backbone colouring. Indeed, the *circular list chromatic number* or *circular choice number* of G may be defined as

$$\mathrm{cch}(G) := \inf\{\mathrm{CBC}_q^\ell(G,G)/q\}.$$

Zhu [18] proved that $\mathrm{cch}(G) \geq \chi^\ell(G) - 1$ for every graph G. This implies that $\mathrm{CBC}_q^\ell(G,G) \geq q \cdot \chi^\ell(G) - q$ for every positive q. In view of this inequality and the fact that $\mathrm{CBC}_q(G,G) = q \cdot \chi(G)$, it is natural to ask whether

$\mathrm{CBC}_q^\ell(G,G)$ can be bounded by a function of $\chi^\ell(G)$. We answer in the positive in Section 2. However, the function we prove is exponential in $\chi^\ell(G)$ – we believe that it is far from optimal, and pose the following problem.

Problem 1. *Let q be a positive integer. What is the minimum function m_q such that for every k-choosable graph G, $\mathrm{CBC}_q^\ell(G,G) \le m_q(k)$? In other words, what is $m_q(k) = \max\{\mathrm{CBC}_q^\ell(G,G) \mid \chi^\ell(G) = k\}$?*

This is closely related to a conjecture of Zhu [18].

Conjecture 2 (Zhu [18])**.** *There is a constant α such that, for every graph G, $\mathrm{cch}(G) \le \alpha \cdot \chi^\ell(G)$.*

Note that if such an α exists then it is at least two, as for any positive integer M, $\chi^\ell(K_{k,m^k}) \le k+1$ and $\mathrm{cch}(K_{k,m^k}) \ge \left(2 - \frac{2k}{m}\right)k$. Conjecture 2 would be implied by the following.

Conjecture 3. *Let q be a positive integer. There is a constant α_q such that, for every graph G, $\mathrm{CBC}_q^\ell(G,G) \le \alpha_q \cdot q \cdot \chi^\ell(G)$.*

As an evidence in support of the latter conjecture, we prove (Theorem 17) that for a graph G and a matching M, $\mathrm{CBC}_2^\ell(G,M) \le 2 \cdot \chi^\ell(G) + 1$ and $\mathrm{CBC}_q^\ell(G,M) \le 2q(1+o(1)\chi^\ell(G)$, where $o(1)$ is a function tending to 0 when $\chi^\ell(G)$ tends to infinity. These bounds generalize to any backbone H in an upper bound in terms of $\chi^\ell(G)$ and $\chi'(H)$, the chromatic index of H: $\mathrm{CBC}_2^\ell(G,H) \le 2^{\chi'(H)}\chi^\ell(G)+1$ and for any $q \ge 3$, $\mathrm{CBC}_q^\ell(G,H) \le 2^{\chi'(H)}q(1+o(1))\chi^\ell(G)$.

In Section 3, we consider list circular backbone colouring of 2-choosable graphs. In [7], Erdős, Rubin, and Taylor characterized such graphs as follows. The *heart* of a graph G is the maximal subgraph in which there is no vertex of degree one. The graph consisting of two vertices connected with three internally vertex-disjoint paths of length i, j and k is $\theta_{i,j,k}$.

Theorem 4 (Erdős *et al.* [7])**.** *A connected graph is 2-choosable if and only if its heart is either a single vertex, an even cycle or $\theta_{2m,2,2}$ for some integer $m \ge 1$.*

Using this characterization, Havet et al. [10] proved that if G is 2-choosable, then $\mathrm{CBC}_q^\ell(G,G) \le 5q/2$. This upper bound was later improved by Norine, Wong, and Zhu [15] to $16q/7$. In other words, $m_q(2) \le 16q/7$. They conjectured that $m_q(2) = 2q$.

Conjecture 5 (Norine, Wong, and Zhu [15])**.** *If G is 2-choosable, then $\mathrm{CBC}_q^\ell(G,G) = 2q$.*

This conjecture is trivial for trees and was proved by Norine [14] for even cycles (and so for all graphs whose heart is an even cycle).

Theorem 6 (Norine [14])**.** *If C is an even cycle, then $\mathrm{CBC}_q^\ell(C,C) = 2q$ for all q.*

Norine, Wong, and Zhu [15] also verified Conjecture 5 for $K_{2,3} = \theta_{2,2,2}$.

Note that Conjecture 5 implies that if G is 2-choosable, then $\mathrm{CBC}_q^\ell(G,H) = 2q$ for any non-empty subgraph H of G. We prove that this is the case if H is a matching (Theorem 28). We also show that if H does not contain the heart of G, then $\mathrm{CBC}_q^\ell(G,H) \le 2q+1$.

1.2 Edge-weighted backbones

In Section 4 we generalize the idea of a backbone colouring by giving every edge e a positive integer weight $w(e)$ dictating the minimum distance between the colours of its endpoints.

Formally, a colouring c of G of G is *w-coherent* if $|c(u) - c(v)| \ge w(uv)$ for every edge uv of G. A graph G is *w-coherent k-choosable* if, given any list assignment L such that $|L(v)| \ge k$ for each vertex v of G, there is a w-coherent L-colouring.

Similarly, a $\mathbb{Z}_p$-colouring c of G is *w-circular* if $|c(u)-c(v)|_p \geq w(uv)$ for every edge uv of G. A graph G is *w-circular k-choosable* if, given any $\mathbb{Z}_p$-list assignment L such that $|L(v)| \geq k$ for each vertex v of G, there is a w-circular L-colouring. Clearly, every w-circular colouring is a w-coherent colouring, and every w-coherent k-colouring induces a w-circular $(k+\max_{v\in V} w(v)-1)$-circular colouring.

The *w-degree* of a vertex v is $d_w(v) = \sum_{e\ni v}(2w(e)-1)$. The *maximum w-degree* of G, denoted $\Delta_w(G)$, is $\max\{w(v) \mid v \in V(G)\}$. Clearly, using the (one-pass) greedy algorithm, every graph is w-circular $(\Delta_w(G)+1)$-choosable.

The *w-semi-degree* of a vertex v is $\tilde{d_w}(v) = \sum_{e\ni v} w(e)$, and the maximum semi-degree of a graph G is $\tilde{\Delta}_w(G) = \max\{\tilde{d_w}(v) \mid v \in V(G)\}$. Using the many-passes greedy algorithm, McDiarmid [12] proved that every graph G is w-coherent $(\tilde{\Delta}_w(G)+1)$-choosable. Therefore every graph G is w-circular $(\tilde{\Delta}_w(G)+\max_{v\in V} w(v))$-choosable.

For a digraph D, we denote by $E^+_D(v)$ the set of arcs with v as their tails, and by $E^-_D(v)$ the set of arcs with v as their heads. Similarly, if D is a digraph, its *w-outdegree* is $d^+_w(v) = \sum_{e\in E^+_D(v)}(2w(e)-1)$.

Extending a result of Norine, Wong and Zhu [15] (Theorem 33), which itself extended a famous result of Alon and Tarsi [3] (Theorem 32), we use the Combinatorial Nullstellensatz to prove that if a graph G admits an orientation D with certain property, then for any $\mathbb{Z}_p$-list assignment L such that $|L(v)| \geq d^+_w(v)+1$ for each vertex v, there is a w-circular L-colouring (Lemma 34). In particular, if D is an orientation of G that contains no odd directed cycles, then it has the required property (Theorem 34). From this we deduce that if G is bipartite, then it is w-circular $(M+1)$-choosable for $M = \sum_i (2i-1)\lceil \mathrm{Mad}(G_i)/2\rceil$, where G_i is the graph induced by the edges of weight i.

1.3 Restricted graph classes

Let $\mathcal{G}$ and $\mathcal{H}$ be two graph classes. For any parameter A in among BBC_q, CBC_q and CBC^ℓ_q, we define $A(\mathcal{G},\mathcal{H})$ as the maximum value of $A(G,H)$ over all pairs (G,H) such that $G\in\mathcal{G}$, $H\in\mathcal{H}$ and H is a subgraph of G. Let $\mathcal{P}$, $\mathcal{F}$ and $\mathcal{M}$ be the classes of planar graphs, forests, and matchings respectively.

In Section 5, we consider the case when G is in $\mathcal{P}$ and H is in one of the three classes $\mathcal{P}$, $\mathcal{F}$ and $\mathcal{M}$. Inequality (1) and the Four-Colour Theorem imply $\mathrm{BBC}_q(\mathcal{P},\mathcal{P}) \leq 3q+1$ and $\mathrm{CBC}_q(\mathcal{P},\mathcal{P}) \leq 4q$. Moreover, $\mathrm{BBC}_q(G,G) = 3q+1$ and $\mathrm{CBC}_q(G,G) = 4q$, for any 4-chromatic planar graph G. Thus

$$\mathrm{BBC}_q(\mathcal{P},\mathcal{P}) = 3q+1 \quad \text{and} \quad \mathrm{CBC}_q(\mathcal{P},\mathcal{P}) = 4q.$$

Extending the celebrated planar 5-choosability proof due to Thomassen [17], Havet et al. [10] proved the that if G is planar, then $\mathrm{CBC}^\ell_q(G,G) \leq 8q-3$. On the other hand, they constructed planar graphs G for which $\mathrm{CBC}^\ell_q(G,G) \geq 6q-2$. Therefore,

$$6q-2 \leq \mathrm{CBC}^\ell_q(\mathcal{P},\mathcal{P}) \leq 8q-3.$$

But the exact value of $\mathrm{CBC}^\ell_q(\mathcal{P},\mathcal{P})$ is still unknown.

Problem 7. *What is the exact value of* $\mathrm{CBC}^\ell_q(\mathcal{P},\mathcal{P})$ *?*

This problem is very closely related to the one posed by Mohar [13].

Problem 8 (Mohar [13])**.** *What is the best upper bound on circular choosability for planar graphs?*

Clearly, $\mathrm{BBC}_q(\mathcal{P},\mathcal{M}) \leq \mathrm{BBC}_q(\mathcal{P},\mathcal{F}) \leq \mathrm{BBC}_q(\mathcal{P},\mathcal{P}) \leq 3q+1$. However, better upper bounds have been obtained for $\mathrm{BBC}_q(\mathcal{P},\mathcal{M})$ and $\mathrm{BBC}_q(\mathcal{P},\mathcal{F})$. Havet et al. [11] proved that for $q\geq 4$, $\mathrm{BBC}_q(\mathcal{P},\mathcal{F}) = q+6$. They also proved that $\mathrm{BBC}_3(\mathcal{P},\mathcal{F}) \leq 9$, and conjectured the following.

Conjecture 9 (Havet et al. [11])**.** $\mathrm{BBC}_3(\mathcal{P},\mathcal{F}) = 8$.

Inequality 1 and the Four-Colour Theorem imply that $\mathrm{BBC}_2(\mathcal{P},\mathcal{F}) \leq 7$. Broersma et al. [5] showed $\mathrm{BBC}_2(\mathcal{P},\mathcal{F}) \geq 6$ and conjectured the following.

Conjecture 10 (Broersma et al. [5]). $\mathrm{BBC}_2(\mathcal{P},\mathcal{F}) = 6$.

Havet et al. [11] showed that Conjecture 10 implies Conjecture 9.

They also proved that $\mathrm{CBC}_q(\mathcal{P},\mathcal{F}) \leq 2q+4$ and conjectured that this upper bound can be reduced by at least one.

Conjecture 11. $\mathrm{CBC}_q(\mathcal{P},\mathcal{F}) \leq 2q+3$.

It might even be possible that the bound $2q+3$ is not optimal.

Regarding matching backbones, Broersma et al. [6] proved that for $\mathrm{BBC}_q(\mathcal{P},\mathcal{M}) = q+3$ for all $q \geq 3$ and conjecture that the same holds for $q = 2$.

Conjecture 12. $\mathrm{BBC}_2(\mathcal{P},\mathcal{M}) = 5$.

By Equation (2), the fact that $\mathrm{BBC}_q(\mathcal{P},\mathcal{M}) \leq q+3$ implies that $\mathrm{CBC}_q(\mathcal{P},\mathcal{M}) \leq 2q+2$. The missing case $q = 2$ of this inequality was proved by Broersma et al. [6].

Proposition 13 (Broersma et al. [6]). $\mathrm{CBC}_q(\mathcal{P},\mathcal{M}) \leq 2q+2$.

They also show an example for which the bound is attained for $q = 2$, so $\mathrm{CBC}_2(\mathcal{P},\mathcal{M}) = 6$. For larger value of q, it is sitll open if the bound $2q+2$ is best possible.

Problem 14. *Is is true that* $\mathrm{CBC}_q(\mathcal{P},\mathcal{M}) \leq 2q+1$, *for all* $q \geq 3$ *?*

We also believe that Conjecture 11 and Proposition 13 extend to list circular backbone colouring.

Conjecture 15. *(i)* $\mathrm{CBC}^{\ell}_q(\mathcal{P},\mathcal{F}) \leq 2q+3$;

(ii) $\mathrm{CBC}^{\ell}_q(\mathcal{P},\mathcal{M}) \leq 2q+2$.

In support of both parts of this conjecture, we derive from our result of Section 4, that if G is a bipartite planar graph G and F a forest in G, $\mathrm{CBC}^{\ell}_q(G,F) \leq 2q+2$. Observe that by (3), we have $\mathrm{CBC}_q(G,F) \leq 2q$. We also show that $\mathrm{CBC}^{\ell}_q(\mathcal{P},\mathcal{M}) \leq 2q+3$ (Theorem 42). Our proof also uses the approach of Thomassen's 5-choosability proof [17].

2 Bounding $\mathrm{CBC}_q(G,H)$ and $\mathrm{CBC}^{\ell}_q(G,H)$in terms of $\chi(G)$ and $\chi^{\ell}(G)$

Theorem 16. *Let* $M_q(k) = \frac{4(k^2+1)^2}{(2q-1)\log_2 e^2} \cdot 2^{2k} + 1$. *Then* $\mathrm{CBC}^{\ell}_q(G,H) \leq M_q(\chi^{\ell}(G))$ *for any subgraph* H *of* G.

Proof. Because $\mathrm{CBC}^{\ell}_q(G,H) \leq \mathrm{CBC}^{\ell}_q(G,G)$ for any graph G, it is sufficient to prove it for $\mathrm{CBC}^{\ell}_q(G,G)$. Let G be a graph such that $\mathrm{CBC}^{\ell}_q(G,G) > M_q(k)$. Let H be the smallest subgraph of G such that $\mathrm{CBC}^{\ell}_q(H,H) > M_q(k)$. Then $\delta(H) > \frac{M_q(k)-1}{2q-1} = \frac{4(k^2+1)^2}{\log_2 e^2} \cdot 2^{2k}$. But Alon [1] proved that if $\delta(G) > \frac{4(k^2+1)^2}{\log_2 e^2} \cdot 2^{2k}$, then $\chi^{\ell}(G) > k$. Therefore $\chi^{\ell}(H) \geq k$ and so $\chi^{\ell}(G) \geq k$. □

We believe the bound $M_q(k)$ of Theorem 16 is far from being be tight. In particular, when H is a matching, we can get an upper bound that is a lot smaller.

Theorem 17. *Let G be a graph and M a matching in G. Then*

(i) $\mathrm{CBC}^{\ell}_2(G,M) \leq 2 \cdot \chi^{\ell}(G) + 1$,

(ii) for any $q \geq 3$ *and any* $\alpha > 1$, *there is* $\beta_q = \beta_q(\alpha)$ *such that* $\mathrm{CBC}^{\ell}_q(G,M) \leq 2q \cdot \alpha \cdot \chi^{\ell}(G) + \beta_q$.

In order to prove this theorem, we need some definitions. Let A and B be two sets of $\mathbb{N}$ or $\mathbb{Z}_p$. We say that A and B *q-interfere* if there exists $a \in A$ and $b \in B$ such that $a \neq b$ and $|a-b| < q$.

For any pair of non-negative integers, we define $f_q(a,b)$ to be the smallest integer m such that for any two subsets of $\mathbb{N}$ of cardinality at least m, there exist two non-interfering sets $A' \subset A$ and $B' \subset B$ such that $|A'| = a$, $|B'| = b$. Similarly, let $g_q(a,b)$ be the smallest integer m such that for any p and any two subsets of $\mathbb{Z}_p$ of cardinality at least m, there exist two non-interfering sets $A' \subset A$ and $B' \subset B$ such that $|A'| = a$, $|B'| = b$. These two functions are closely related.

Proposition 18.

$$f_q(a,b) \leq g_q(a,b) \leq f_q(a,b) + q - 1.$$

Proof. If two sets q-interfere in $\mathbb{N}$, then they also 2-interfere in $\mathbb{Z}_r$ (for r sufficiently large). So $f_q(a,b) \leq g_q(a,b)$. Now let C and D be two sets in $\mathbb{Z}_p$ of cardinality $f_q(a,b)+q-1$, and consider $A = C \setminus \{1,\dots,q-1\}$ and $B = D \setminus \{1,\dots,q-1\}$. Then $|A| = |B| = f_q(a,b)$, so there exist two sets $A' \subset A$ and $B' \subset B$ of cardinality a and b respectively which do not q-interfere in $\mathbb{N}$. But by definition of A and B, $A' \subset C$, $B' \subset D$ and they cannot q-interfere in $\mathbb{Z}_p$. □

Let us define h_q by $h_q(k) = g_q(k,k)$. The function h_q is useful for our purpose as shown by the following lemma.

Lemma 19. *Let G be a graph, H a subgraph of G and M a matching in H. Then*

$$\mathrm{CBC}_q^{\ell}(G,H) \leq h_q(\mathrm{CBC}_q^{\ell}(G,H-M)).$$

In particular,

$$\mathrm{CBC}_q^{\ell}(G,M) \leq h_q(\chi^{\ell}(G)).$$

Proof. Let $k = \chi^{\ell}(G)$. Let L be a $h_q(k)$-list assigment in Z_p. By definition of h_q, for every edge $uv \in M$, we can find two k-sets $L'(u) \subset L(u)$ and $L'(v) \subset L(v)$ that do not 2-interfere. Since G is k-choosable, there is an L'-colouring c of G.

We now show that c is a q-backbone colouring of (G,M). First, for any edge $xy \in E(G)$, $c(x) \neq c(y)$. For any edge $uv \in M$, $c(u) \neq c(v)$. In addition, since $L'(u)$ and $L'(v)$ do not q-interfere, $|c(u)-c(v)| \geq q$. □

Notation For two integers a and b, we denote by $[a,b]^p$, the set $\{a,a+1,\dots,b\}$, where the numbers are modulo p. For every integer a, we denote by $[a]_q^p$ the set $[a-q+1,a+q-1]^p$. Very often p is either clear from the context or implicit and we omit the supercript p.

Lemma 20. *(i)* $f_2(a,b) \leq a+b$.

(iii) For $q \geq 3$ let $i^* = i^*(q)$ be the smallest integer such that $(q-1)i^* < 2^{i^*+1} - 2$, and set $R(q) = (2q-3)2^{i^*} - 2$. Then $f_q(a,b) \leq 2(a+b) + R(q)$.

Proof. (i) We prove the result by induction on $a+b$, the result holding trivially when $a=0$ or $b=0$. Let x be the minimum of $A \cup B$.

Suppose first $x \in A \cap B$. Let $C = A \setminus \{x,x+1\}$, and $D = B \setminus \{x,x+1\}$. Then $|C| = |D| = a+b-2$, so by the induction hypothesis, there exist two non-2-interfering sets $C' \subset C$ and $D' \subset D$ such that $|C'| = a-1$, $|D'| = b-1$. Setting $A' = C' \cup \{x\}$ and $B' = D' \cup \{x\}$, we obtain the desired sets.

Suppose now $x \notin A \cap B$. Then x is only in one of these two sets, say A. Let $C = A \setminus \{x\}$, and $D = B \setminus \{x+1\}$. Then $|C| = |D| = a+b-1$, so by the induction hypothesis, there exist two non-2-interfering sets $C' \subset C$ and $D' \subset D$ such that $|C'| = a-1$, $|D'| = b$. Setting $A' = C' \cup \{x\}$ and $B' = D'$, we obtain the desired sets.

(ii) We prove the result by induction on $a+b$.

Without loss of generality,we may assume $a \le b$. Assume $a < 2^{i^*} - 1$. Then one can choose any a-subset A' of A. It forbids at most $a \times 2q - 2$ elements to be chosen in B. So if $|B|$ is of cardinality at least $b + a + R(q) \ge b + (2q-2)a$, we can find a b-subset B' that does not q-interfere with A'.

Assume now that a and b are both greater or equal to $2^{i^*} - 1$.

Let x be the smallest element of $A \cup B$. Without loss of generality, we may assume that $x \in A$. Consider $C_1 = A \setminus \{x\}$ and $D_1 = B \setminus [x, x+q-1]$. If $|B \cap [x, x+q-1]| \le 2$, then C_1 and D_1 are both of cardinality at least $2(a+b) + R(q) - 2$. Hence by the induction hypothesis, we can find an $(a-1)$-set C' in C_1 and a b-set B' in D_1 which do not q-interfere. Then, $A' = C' \cup \{x\}$ and B' are the desired sets. Hence we may assume that at least three elements of B are in $[x, x+q-1]$.

For any k, let $A(k)$ (resp. $B(k)$) be the set of the k smallest elements in A (resp. B). Consider $D_2 = B \setminus B(3)$ and $C_2 = A \setminus [x, x+2(q-1)]$. If $|A \cap [x, x+2(q-1)]| \le 6$, then C_2 and D_2 are both of cardinality at least $a+b+R(q)-6$. Hence by the induction hypothesis, we can find an a-set A' in C_2 and a $(b-3)$-set D' in D_2 which do not q-interfere. Then, A' and $B' = D' \cup B(3)$ are the desired sets. (they do not interfere because $B(3) \subset [x, x+q-1]$. Hence we may assume that $A(7) \subset [x, x+2(q-1)]$.

And so on for $2 \le i \le i^*$. If i is odd, consider $C_i = A \setminus A(2^i - 1)$ and $D_i = B \setminus [x, x+i(q-1)]$. If $|B \cap [x, x+i(q-1)]| \le 2^{i+1} - 2$, then C_i and D_i are both of cardinality at least $2(a+b) + R(q) - 2^{i+1} - 2$. Hence by the induction hypothesis, we can find an $(a - 2^i + 1)$-set C' in C_i and a b-set B' in D_i which do not q-interfere. Then, $A' = C' \cup A(2^i - 1)$ and B' are the desired sets. Hence we may assume that $B(2^{i+1} - 1) \subset [x, x+i(q-1)]$. If i is odd, consider $B_i = B \setminus B(2^i - 1)$ and $C_i = B \setminus [x, x+i(q-1)]$. If $|A \cap [x, x+i(q-1)]| \le 2^{i+1} - 2$, then C_i and D_i are both of cardinality at least $2(a+b) + R(q) - 2^{i+1} - 2$. Hence by the induction hypothesis, we can find an a-set A' in C_i and a $(b - 2^i + 1)$-set B' in D_i which do not q-interfere. Then, A' and $B' = D' \cup B(2^i - 1)$ are the desired sets. Hence we may assume that $A(2^{i+1} - 1) \subset [x, x+i(q-1)]$.

So, depending on the parity of i^*, $[x, x+i(q-1)]$ contains at least $2^{i^*+1} - 1$ vertices of either A or B. But this is impossible, because $[x, x+i(q-1)]$ has cardinality less than $2^{i^*+1} - 1$ by definition of i^*. □

Corollary 21. *(i)* $g_2(a,b) \le a+b+1$.

(ii) for any $q \ge 3$, $f_q(a,b) \le a+b+R(q)+q-1$.

Remark 22. $a+b+1$ may not be replaced by $a+b$, even if $a = b$, in Corollary 21 (i). Just consider $a = b$ and Z_{4a} with A the set of even numbers and B the set of odd numbers. One can check that for any pair of two a-sets $A' \subset A$ and $B' \subset B$ 2-interfere.

Theorem 17 derives directly from Corollary 21 and Lemma 19. It can also be generalized to any subgraph H. Indeed in the very same way as Lemma 19, we can prove the following.

Lemma 23. *Let G be a graph, H a subgraph of G and M a matching in H. Then*

$$\mathrm{CBC}_q^{\ell}(G,H) \le h_q(\mathrm{CBC}_q^{\ell}(G, H-M)).$$

Hence by induction, we get the following theorem, where $\chi'(H)$ denotes the chromatic index of H.

Theorem 24. *Let G be a graph and H a subgraph of G and let $q \ge 2$ be an integer. Then* $\mathrm{CBC}_q^{\ell}(G,H) \le 2^{\chi'(H)} q(1 + o(1))\chi^{\ell}(G)$.

Remark 25. In the case $q = 2$, one can get rid of the $o(1)$ term and show that $\mathrm{CBC}_2^{\ell}(G,H) \le 2^{\chi'(H)}\chi^{\ell}(G) + 1$. It suffices to observe that we can remove one fixed colour so that the colours behave like on $\mathbb{N}$. Then for each matching M of a proper edge-colouring, we must divide the list by 2 so that there is no 2-interference between vertices of linked by an edge in M.

3 2-choosable graphs

In this section we study list backbone colouring of 2-choosable graphs.

Let t and q be two positive integers. The (t,q)-*kernel* of (G,H) is the maximal subset K of $V(G)$ such that $d_{G[K]}(v)+(2q-2)d_{H[K]}(v) \geq t$ for every vertex $v \in K$. Observe that if $t \geq 2q$, then the graph $G[K]$ is a subgraph of the heart of G.

By convention, $\mathrm{CBC}_q^{\ell}(N,N)=1$ if N is the null graph.

Lemma 26. *Let t and q be two positive integers, and let G be a graph and M a matching in G, and let K be its (t,q)-kernel. Then $\mathrm{CBC}_q^{\ell}(G,M) \leq t$ if and only if $\mathrm{CBC}_q^{\ell}(G[K],M[K]) \leq t$.*

Proof. Let L be an t-list assignment on G in Z_r. It is easy to see that for any subgraph G' of G, if there is a vertex v such that $d_{G'}(v)+(2q-2)d_{H'}(v) < t$, where $H'=H[V(G')]$, then any q-backbone L-colouring of $(G'-v,H'-v)$ can be extended into a q-backbone L-colouring of (G',H'). The result follows. □

Lemma 26 and Theorem 6 immediately imply the following.

Corollary 27. *Let G be a graph and H a nonempty subgraph of G. If the heart of G is a single vertex or an even cycle, then $\mathrm{CBC}_q^{\ell}(G,M)=2q$.*

Theorem 28. *If G is a 2-choosable graph and M a matching, then $\mathrm{CBC}_q^{\ell}(G,M)=2q$.*

Proof. Clearly, it is enough to prove it for connected graphs. So we may assume that G is connected.

Let H be the heart of G, and let $N=M[V(H)]$. Observe that if H is a $\theta_{2m,2,2}$ for some integer $m \geq 1$, with u and v its two vertices of degree three, then $G-\{u,v\}$ has three connected components. Hence H cannot have a perfect matching, and so at least two vertices of H are not matched in N, and so there is a vertex v of H such that $d_H(v)+(2q-2)d_N(v)=d_H(v)<2q$.

It follows that the $(2q,q)$-kernel of (G,M) is either the empty set, or a set K such that $G[K]$ is an even cycle and $M[K]$ a perfect matching of $G[K]$. Hence, by Theorem 6, $\mathrm{CBC}_q^{\ell}(G[K],M[K]) \leq 2q$, and so by Lemma 26, $\mathrm{CBC}_q^{\ell}(G,M) \leq 2q$. □

Remark 29. In the above prove, Theorem 6 is not necessarily required as it is easy to see that $\mathrm{CBC}_q^{\ell}(G,M) \leq 2q$ for an even cycle G and a matching M.

Theorem 30. *If G is a 2-choosable graph and H is a subgraph of G not containing its heart, then $\mathrm{CBC}_q^{\ell}(G,H) \leq 2q+1$.*

Proof. Again it is enough to prove the statement for G connected. If H does not contain the heart of G, then one easily sees that the $(2q+1,q)$-kernel of (G,H) is either the empty graph or an even cycle. Therefore Theorem 6 and Lemma 26 yield the result. □

4 Weighted circular list colouring via the Combinatorial Nullstellensatz

In this section we shall use the following theorem, called the Combinatorial Nullstellensatz, to establish some results on circular backbone colouring.

Theorem 31. *Let F be an arbitrary field, and let $P=P(x_1,\dots,x_n)$ be a polynomial in $F[x_1,\dots,x_n]$. Suppose that the degree $deg(P)$ of P is $\sum_{i=1}^n t_i$, where each t_i is a non-negative integer, and suppose the coefficient of $\prod_{i=1}^n x_i^{t_i}$ in P is non-zero. Then, if $S_1,\dots,S_n$ are subsets of F with $|S_i|>t_i$, there are $s_1 \in S_1,\dots,s_n \in S_n$ so that*

$$P(s_1,\dots,s_n) \neq 0.$$

Let G be a graph with vertex set $\{v_1,\dots,v_n\}$. The *graph polynomial* f_G of G is defined by $f_G(x_1,\dots,x_n) = \prod\{(x_i-x_j) \mid i<j, v_iv_j \in E(G)\}$. Alon and Tarsi [3] applied the Combinatorial Nullstellensatz to the graph polynomial to obtain the following theorem on choosability.

Theorem 32 (Alon and Tarsi [3])**.** *Let D be an orientation of a graph G, and let $ee(D)$ and $oe(D)$ respectively the set of even and odd eulerian spanning subdigraphs of D. If $ee(D) \neq oe(D)$, then for any list assignment L such that $|L(v)| \geq d^+(v)+1$ for all v, the graph G is L-choosable.*

Let G be a graph with vertex set $\{v_1,v_2,\dots,v_n\}$ and let p and q be two positives integers. Its *(p,q)-circular polynomial* of G is

$$CP^G_{p,q}(x_1,x_2,\dots,x_n) = \prod_{\substack{v_jv_{j'}\in E(G)\\ j<j'}} \prod_{k=-q+1}^{q-1} (x_j - \exp(2\pi ik/p)x_{j'}).$$

Let $\phi : \{0,\dots,p-1\} \to \mathbb{C}$ be the function defined by $\phi(k) = \exp(2\pi ik/p)$. It is obvious that a mapping $c : V(G)\{0,\dots,p-1\}$ is a (p,q)-colouring of G if and only if

$$CP^G_{p,q}(c(v_1),c(v_2),\dots,c(v_n)) \neq 0.$$

Applying the Combinatorial Nullstellensatz to the (p,q)-circular polynomials, Norine, Wong and Zhu [15] established a generalization of Theorem 32, from which they deduce the following.

Theorem 33 (Norine, Wong and Zhu [15])**.** *Suppose G is a graph and D is an orientation of G which contains no odd directed cycles. Let L be a $\mathbb{Z}_p$-list assignment for G such that $|L(v)| = (2q-1)d^+_D(v)+1$ for each vertex v. Then G is L-(p,q)-colourable.*

In this section, we shall extend Theorem 33 to w-circular colouring. We will prove the following theorem.

Theorem 34. *Suppose G is a graph with a positive integer edge weighting w. Suppose moreover that D is an orientation of G which contains no odd directed cycles. If L is a $\mathbb{Z}_p$-list assignment for G such that $|L(v)| = d^+_w(v)+1$ for each vertex v, then G is w-circularly L-colourable.*

Let G be a graph with vertex set $\{v_1,v_2,\dots,v_n\}$, w an edge weight of G, and p a positive integer. The *(p,w)-circular polynomial of G* is

$$CP^G_{p,w}(x_1,x_2,\dots,x_n) = \prod_{\substack{v_jv_{j'}\in E(G)\\ j<j'}} \prod_{k=-w(v_jv_{j'})+1}^{w(v_jv_{j'})-1} (x_j - \exp(2\pi ik/p)x_{j'}).$$

Observe that when $w(e) = q$ for all $e \in E(G)$, then $CP^G_{p,w} = CP^G_{p,q}$. It is obvious that a mapping $c : V \to \mathbb{Z}_p$ is a w-circular colouring of G if and only if

$$CP^G_{p,w}(c(v_1),c(v_2),\dots,c(v_n)) \neq 0.$$

Let $\gamma : \mathbb{Z}_p \to \mathbb{C}$ be defined as $\gamma(l) = \exp(2\pi il/p)$ for $l \in \mathbb{Z}_p$. Thus with $S_i = \{\gamma(a) \mid a \in L(v_i)\}$, the graph G is w-circularly L-colourable if and only if there exist $s_1 \in S_1, s_2 \in S_2,\dots,s_n \in S_n$ such that $CP^G_{p,w}(s_1,s_2,\dots,s_n) \neq 0$.

For any orientation D of G is considered, the degree of $CP^G_{p,w}$ is $\sum_{e\in E}(2w(e)-1) = \sum_{j=1}^n d^+_w(v_j)$. Thus by Theorem 31, if there is an orientation D such that the coefficient of the monomial $\prod_{j=1}^n x_j^{d^+_w(v_j)}$ is nonzero, the graph G is w-circularly L-colourable for any $\mathbb{Z}_p$-list assignment L such that $|L(v)| = d^+_w(v)+1$ for all $v \in V(G)$.

Let D be an orientation of G. A non-negative integer mapping $\phi : E(D) \to \mathbb{Z}$ is *w-consistent* if $0 \leq \phi(e) \leq 2w(e)-1$ for all $e \in E(D)$. It is *eulerian* if for every vertex v,

$$\sum_{e\in E^+_D(v)} \phi(e) = \sum_{e\in E^-_D(v)} \phi(e).$$

A mapping that is consistent and eulerian is said to be *nice*. A mapping ϕ is called *even* (respectively, *odd*) if $\sum_{e\in E(D)}\phi(e)$ is even (respectively, odd).

For an arc $(v_j, v_{j'})$ of D of weight q and for $0 \le \phi(e) \le 2q-1$, the coefficient of $x_j^{2q-1-\phi(e)} x_{j'}^{\phi(e)}$ in $\prod_{k=-q+1}^{q-1}(x_j - \exp(2\pi i k/p)x_{j'})$ is equal to

$$\sum_{\substack{J\subseteq\{-q+1,\dots,q-1\}\\ |J|=\phi(e)}} \prod_{j\in J}(-\exp(2\pi i j/p)) = (-1)^{\phi(e)} a_{\phi(e)}(p,q),$$

where

$$a_{\phi(e)}(p,q) = \sum_{\substack{J\subseteq\{-q+1,\dots,q-1\}\\ |J|=\phi(e)}} \prod_{j\in J}\exp(2\pi i j/p).$$

Hence it is easy to see that a w-consistent mapping ϕ makes a contribution to the monomial $\prod_{j=1}^n x_j^{d_w^+(v_j)}$ if and only if it is nice. Moreover, such a mapping contributes for an amount $A(p,w,\phi) = \prod_{e\in E(D)} a_{\phi(e)}(p, w(e))$. Therefore, the coefficient of $\displaystyle\prod_{j=1}^n x_j^{d_w^+(v_j)}$ in $CP_{p,w}^G$ is

$$\sum_{\phi \text{ nice}} \prod_{e\in E(D)} (-1)^{\phi(e)} a_{\phi(e)}(w(e),p) = \sum_{\phi \text{ nice and even}} A(p,w,\phi) - \sum_{\phi \text{ nice and odd}} A(p,w,\phi).$$

Hence, we get the following lemma.

Lemma 35. *Suppose a graph G has an orientation D for which*

$$\sum_{\phi \text{ nice and even}} A(p,w,\phi) \neq \sum_{\phi \text{ nice and odd}} A(p,w,\phi).$$

If L is a $\mathbb{Z}_p$-list assignment for G such that $|L(v)| = d_w^+(v)+1$ for each vertex v, then G is w-circularly L-colourable.

Observe now that each $a_l(q,p)$ is a real because it is equal to its conjugate. We can also see that $a_0(q,p) = a_{2q-1}(q,p) = 1$. A result of Evans and Montgomery [8] implies that if $p \ge 2q$, then the values of $a_l(q,p)$, for fixed q and p, are unimodal. It follows that all values of $a_l(p,q)$ are at least 1 – Norine, Wong and Zhu proved this from first principles for certain values of p and q [15].

Lemma 36. *Let p and q be two positive integers such that $p \ge 2q$. For any $0 \le l \le 2q-1$, $a_l(p,q) \ge 1$.*

From Lemmas 35 and 36, we now derive Theorem 34.

Proof of Theorem 34. By Lemma 35, it suffices to show that

$$\sum_{\phi \text{ nice and even}} A(p,w,\phi) \neq \sum_{\phi \text{ nice and odd}} A(p,w,\phi).$$

For a nice mapping ϕ, let D_ϕ be the multi-digraph obtained from D by replacing each arc $(v_j, v_{j'})$ by $\phi(e)$ parallel arcs from v_j to $v_{j'}$. Since ϕ is eulerian, then D_ϕ is an eulerian multi-digraph. Each directed cycle of D_ϕ corresponds to a directed cycle of D. Since D has no directed cycle of odd length, D_ϕ has no directed cycle of odd length. Thus $|E(D_\phi)|$ is even, i.e. $\sum_{e\in E(D)}\phi(e)$ is even. So D has no odd nice mapping, and so $\sum_{\phi \text{ nice and odd}} A(p,w,\phi) = 0$.

Now Lemma 36 implies that all the $A(p,w,\phi)$ are positive. But there is at least one nice even mapping, namely the one for which $\phi(e) = 0$ for all $e \in E(D)$. So $\sum_{\phi \text{ nice and even}} A(p,w,\phi) > 0$. □

We shall now apply Theorem 34. We need the following simple lemma which was first proved by Hakimi [9]. It also appears independently in several papers [16, 2, 3].

Lemma 37. *A graph G has an orientation D with maximum outdegree at most Δ^+ if and only if* $\mathrm{Mad}(G)/2 \leq \Delta^+$.

For a graph G and an edge weighting w, we define $M(G,w) := \sum_{i\in\mathbb{N}^*}(2i-1)\lceil \mathrm{Mad}(G_i)/2\rceil$, where G_i is the subgraph of G induced by the edges of weight i, for all $i \in \mathbb{N}^*$.

Corollary 38. *Let G be a graph and w an edge-weighting. Then G has an orientation D such that $d^+_w(v) \leq M(G,w)$ for all $v \in V$.*

Proof. It suffices to orient each G_i separately in such a way that the maximum outdegree is at most $\lceil \mathrm{Mad}(G_i)/2\rceil$, which is possible by Lemma 37. The union of these orientations yields the desired one. □

Theorem 34 and Corollary 38 immediately yield the following.

Theorem 39. *Let G be a bipartite graph and w an edge weighting. Then G is w-circular $(M(G,w)+1)$-choosable.*

Proof. By Corollary 38 , G admits an orientation D such that $d^+_w(v) \leq M(G,w)$ for all $v \in V$. Since G is bipartite, it has no odd cycles and so D has no directed odd cycles. Hence by Theorem 34, G is $(M(G,w)+1)$-choosable. □

5 A stronger result for planar graphs

If G is a bipartite planar graph then $\mathrm{Mad}(G) < 4$, and if T is a tree then $\mathrm{Mad}(T) < 2$. Thus, Theorem 39 directly implies the following.

Corollary 40. *If G is a bipartite planar graph and T a spanning tree of G, then* $\mathrm{CBC}^\ell_q(G,T) \leq 2q+2$.

When the planar graph G is no longer required to be bipartite, we can use an acyclic orientation D with maximum outdegree 5. Such an orientation exists because a planar graph is 5-degenerate. Considering such an orientation, we derive the following, which corresponds to greedy colour according to a linear ordering extending D.

Corollary 41. *Let G be a planar graph and H be a subgraph in G. Then* $\mathrm{CBC}^\ell_q(G,H) \leq (2q-2)\Delta(H)+6$.

Proof. Let w be the edge-weight defined by $w(e) = q$ if $e \in E(H)$ and $w(e) = 1$ if $e \in E(G)\setminus E(H)$. Then for the orientation D with maximum outdegree 5, we have $\Delta^+_w(D) \leq (2q-2)\Delta(H)+5$. Since D is acyclic, it contains no odd directed cycles, and so by Theorem 34, (G,w) is circular $((2q-2)\Delta(H)+6)$-choosable. □

In particular, if G is a planar and M a matching, $\mathrm{CBC}^\ell_q(G,M) \leq 2q+4$. Using a different technique, we now prove a slightly weaker bound.

Theorem 42. *Let G be a planar graph and M be a matching in G. Then* $\mathrm{CBC}^\ell_q(G,M) \leq 2q+3$.

We borrow the approach of Thomassen's 5-choosability proof [17], and actually establish a slightly stronger result.

Theorem 43. *Let G be a near triangulation with outer cycle C, M a matching in G and let L be a list assignment such that:*

- $|L(v)| \geq 3$ *for all* $v \in C\setminus M$;
- $|L(v)| \geq 2q+1$ *for all* $v \in C\cap M$;

- $|L(v)| \geq 2q+3$ *otherwise.*

Then any valid precolouring of two adjacent vertices of C can be extended to a circular q-backbone L-colouring of (G,M).

To prove this theorem, we need the following lemma.

Lemma 44. *Let q be a positive integer and let S and T be two sets of size* $2q$ *and* $2q+3$ *respectively in* $\mathbb{Z}_p$. *Then there exist two elements a and b of S such that* $|T \setminus ([a]_q \cup [b]_q)| \geq 3$.

Proof. The result holds trivially when $q = 1$.

Let us now prove the case $q = 2$. If two elements a and b of S are consecutive, then $[a]_2 \cup [b]_2 = 4$, and we trivially have the result. So we may assume that the elements of S are pairwise non-consecutive. Let $1 \leq s_1 \leq s_2 \leq s_3 \leq s_4 \leq p$ be the elements of S and set $I = [s_1]_2 \cup [s_2]_2$ and $J = [s_3]_2 \cup [s_4]_2$. Because the elements of S are pairwise non-consecutive, $|I \cap J| \leq 2$. But,

$$|I \cap T| + |J \cap T| = |(I \cup J) \cap T| + |(I \cap J) \cap T| \leq |T| + |I \cap J| \leq 9.$$

Hence either $I \cap T$ or $J \cap T$ has size at most 4. Therefore either s_1 and s_2 or s_3 and s_4 are the desired elements of S.

Suppose now that $q \geq 3$. We prove the result by induction on p.

If two elements a and b of S are consecutive (in particular if $p \leq 4q-1$, then $[a]_q \cup [b]_q = 2q$, and we trivially have the result. Thus we may assume that the elements of S are pairwise non-consecutive.

Assume now that that an element, say p, is not in $S \cup T$. Then apply induction on Z_{p-1} with the same S and T. It is easy to see that the two elements a and b such that $|T \setminus ([a]_q^{p-1} \cup [b]_q^{p-1})| \geq 3$ also satisfy $|T \setminus ([a]_q^p \cup [b]_q^p)| \geq 3$.

Let $1 \leq s_1 \leq s_2 \leq \cdots \leq s_{2q} \leq p$ be the vertices of S. For $1 \leq i \leq 2q-1$, $I_i = [s_i, s_{i+1}]$ and let $I_{2q} = [s_{2q}, r] \cup [1, s_1]$. Set $J_i = I_i \setminus S$ for each i. By the above arguments, each J_i contains at least one vertex of T, so at most 3 vertices are in $T \cap S$. Moreover a vertex of T is in exactly one I_i if it is in $T \setminus S$ and in exactly two I_i if it is in $T \cap S$. Hence if $q \geq 4$, there must be an I_i such that $|T \cap I_i| = 1$. Then s_i and s_{i+1} satisfy $|T \setminus ([s_i]_q \cup [s_{i+1}]_q)| \geq 3$. If $q = 3$, then either there is an I_i such that $|T \cap I_i| = 1$ and we get the result, or $S \cap T$ is one of the two sets $\{s_1, s_3, s_5\}$ and $\{s_2, s_4, s_6\}$. In that case, one can check that $|T \setminus ([s_1]_q \cup [s_2]_q)| \geq 3$. □

Proof of Theorem 43. The proof is by induction on the number of vertices n. The result holds if G is a triangle since there is at least $3-2=1$ choice to colour the last vertex v if it is not matched, and at least $2q+1-2q=1$ choice to colour the last vertex v if it is matched. Assume now that the result is true for every near triangulation with at most $n-1 \geq 3$ vertices, and let G be a near triangulation with n vertices. We let $u_1u_2\dots u_k$ be the outer cycle of G, and u_1 and u_2 be the two precoloured vertices with respective colours c_1 and c_2.

Case 1: *G has a chord* u_iu_j *with* $i<j$. We use the induction hypothesis on the near triangulation whose outer cycle is $u_1u_2\dots u_iu_ju_{j+1},\dots,u_k,u_1$. Next we use the induction hypothesis on the near triangulation whose outer cycle is $u_iu_{i+1}\dots u_ju_i$, the two precoloured vertices being u_i and u_j. The result follows easily.

Case 2: *G has no chord.* Let $v_1,\dots,v_d$ be the neighbours of u_k that do not belong to C. Without loss of generality, we can assume that $u_{k-1}v_1v_2\dots v_du_1$ is a path. Set $G' = G - u_k$, noting that G' is a near-triangulation.

We shall distinguish four cases:

- Subcase 1: $u_ku_1 \in M$. Since $L(u_k) \geq 2q+1$, there exist two colours a and b in $L(u_k) \setminus [c_1]_q$. We define the list assignment L' of G' by $L'(v) := L(v)$ if $v \notin \{v_1,v_2,\dots,v_d\}$ and $L'(v) := L(v) \setminus \{a,b\}$ otherwise. Then $|L'(v_i)| \geq 2q+3-2 = 2q+1$ for each i. Thus we can apply the induction hypothesis to G' and L'. Now we complete the colouring of G by colouring u_k with a if $c(u_{k-1}) \neq a$ and with b otherwise.

- Subcase 2: $u_{k-1}u_k \in M$. Since $L(u_k) \geq 2q+1$, there exist two colours a and b in $L(u_k) \setminus \{c_1\}$ such that $[a]_q \cap [b]_q = \emptyset$. We define the list assignment L' of G' by $L'(v) := L(v)$ if $v \notin \{v_1, v_2, \dots, v_d\}$ and $L'(v) := L(v) \setminus \{a,b\}$ otherwise. Again $|L'(v_i)| \geq 2q+3-2 = 2q+1$ for each i. Thus we can apply the induction hypothesis to G' and L'. Now we complete the colouring of G by colouring u_k with a if $c(u_{k-1}) \notin [a]_q$ and with b otherwise.

- Subcase 3: u_k is matched to v_j, $1 \leq j \leq d$. Since $L(u_k) \geq 2q+1$, by Lemma 44, there exist two colours a and b in $L(u_k) \setminus \{c_1\}$ such that $|L(v_j) \setminus ([a]_q \cup [b]_q)| \geq 3$. We define the list assignment L' of G' by $L'(v) := L(v)$ if $v \notin \{v_1, v_2, \dots, v_d\}$, $L'(v_j) := L(v_j) \setminus ([a]_q \cup [b]_q)$ and and $L'(v) := L(v) \setminus \{a,b\}$ otherwise. Then $|L'(v_i)| \geq 2q+3-2 = 2q+1$ for each $i \neq j$. Thus we can apply the induction hypothesis to G' and L'. Now we complete the colouring of G by colouring u_k with a if $c(u_{k-1}) \neq a$ and with b otherwise.

- Subcase 4: u_k is not matched. Since $L(u_k) \geq 3$, there exist two colours a and b in $L(u_k) \setminus \{c_1\}$. We define the list assignment L' of G' by $L'(v) := L(v)$ if $v \notin \{v_1, v_2, \dots, v_d\}$ and $L'(v) := L(v) \setminus \{a,b\}$ otherwise. Then $|L'(v_i)| \geq 2q+3-2 = 2q+1$ for each i. Thus we can apply the induction hypothesis to G' and L'. Now we complete the colouring of G by colouring u_k with a if $c(u_{k-1}) \neq a$ and with b otherwise. □

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